Sydney Boys High School

MOORE PARK
SURRY HILLS

DECEMBER 2003

HSC Assessment Task #1

YEAR 11

Mathematics

General Instructions

- Reading time – 5 minutes.
- Working time 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for careless or badly arranged work.
- Start each question in a separate answer booklet.

Total Marks - 80 Marks

- Attempt Questions 1 to 5
- All questions are of equal value.

Examiner: *R. Boros*

Question 1: (16 marks) Marks

(a) Evaluate $\log_p 18$ given that $\log_p 3 = 0 \cdot 4771$ and $\log_p 2 = 0 \cdot 3010$. 2

(b) Write a single logarithm for $\log x - \log y + 2\log z$. 1

(c) For what value of n is the sum of n terms of 12 + 15 + 18 + … equal to 441? 2

(d) Evaluate $\sum_{n=3}^{13} 2^n$ 2

(e) One card is drawn out from a set of cards numbered 1 to 20. Find the probability of drawing out an even number or a number less than 8. 2

(f) When 2 regular dice are thrown and the total on these dice are counted, find the probability of scoring a total greater than 7. 2

(g) A plant has a probability $0 \cdot 7$ of producing a variegated leaf. If 3 plants are grown, find the probability of producing no plants with variegated leaves. 3

(h) A coin is tossed n times. Find an expression for the probability of throwing at least 1 tail. 2

Question 2: (16marks) START A NEW BOOKLET Marks

(a) Simplify $\dfrac{(x^{m+1})^2 \times (x^3)^{n+1}}{x^{5m}}$. 2

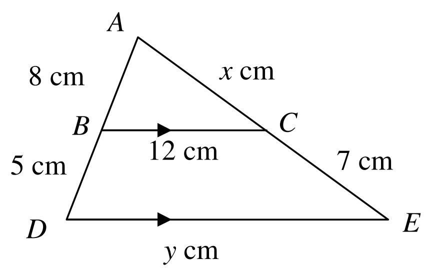
(b) Solve for x: $2^{x-1} = \dfrac{\sqrt{2}}{32}$ 2

(c) Write in simplest form: $\dfrac{2^{n+2}+8}{2^{2n}+2^{n+1}}$ 2

(d) Show that the points $A(6a,-2b)$, $B(2a,0)$ and $C(0,b)$ are collinear. 2

(e) Prove that the points $A(3,5)$, $B(4,4)$, $C(1,1)$ and $D(0,2)$ are the vertices of a rectangle. 4

(f) Prove that $\Delta ABC \;|||\; \Delta ADE$. Hence find the values of x and y. 4



Question 3: (16 marks) START A NEW BOOKLET Marks

(a) Find $\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}$ 2

(b) (i) Find the gradient of the tangent to the curve $y = x^2 + 2x + 1$ at the point (x, y).

(ii) Hence find the gradient of the tangent at the point $(\frac{1}{2}, 2\frac{1}{4})$. 3

(iii) Find the angle which the tangent in (ii) makes with the positive direction of the x axis.

(c) Find the first derivative of:

(i) $y = \frac{-7}{x+1}$

(ii) $y = (x^2 + x)^3$ 5

(iii) $y = \frac{1}{\sqrt{3x^2 + 4}}$

(d) Find the gradient of the normal to the curve $y = 5x\sqrt{4 - x}$ at the point (3, 15) 3

(e) Find the maximum value of the function $y = x^2 - 4x + 3$ in the domain $1 \le x \le 4$. 3

Question 4: (16 marks) START A NEW BOOKLET Marks

(a) For the curve $y = 2x^3 - 3x^2 - 12x + 2$:

(i) Find all stationary points.

(ii) Determine the nature of the stationary points. 9

(iii) Find any points of inflexion.

(iv) Sketch the curve.

(b) Show that $y = \frac{5}{x}$ is always a decreasing function for all real $x \neq 0$. 2

(c) Draw a neat sketch of a continuous curve $y = f(x)$ which has the following features:

$f'(x) < 0$ for $0 \leq x < 3$

$f'(3) = 0$

$f'(x) > 0$ for $3 < x < 7$ 3

$f'(7) = 0$ and

$f'(x) > 0$ for $7 < x \leq 10$.

(d) For a certain curve $y'' = x^2(x-1)^3(x-3)$, for what values of x is the curve concave up? 2

Question 5: (16 marks) START A NEW BOOKLET Marks

(a) Solve for x (correct to 2 decimal places): $2^x = 3^{x-1}$. 2

(b) If $x^2 + y^2 = 7xy$, show that $\log(x+y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y$. 2

(c) A ball is dropped from a height of 1 metre and bounces to $\frac{2}{3}$ of its height on each bounce. What is the total distance travelled by the ball? 3

(d) A sum of \$3 000 is invested at the beginning of each year in a superannuation fund. At the end of 35 years, how much money is available if the money invested earns interest at the rate of 6% per annum (compounded annually). 4

(e) A sum of \$75 000 is borrowed at an interest rate of 12% per annum, monthly reducible. If the money is repaid at regular monthly intervals over 10 years, find the amount of each payment. 5

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SAMPLE SOLUTIONS

16

Question ①

(a) $\log_p 18 = \log_p 3^2 + \log_p 2$

$= 2\log_p 3 + \log_p 2$

$= 1.2552.$ **2**

(1) (b) $\log x - \log y + 2\log z$

$= \log\left(\frac{xz^2}{y}\right).$

(c) $S_n = \frac{n}{2}(2a+(n-1)d)$

$\therefore 441 = \frac{n}{2}[24+(n-1)3]$

$\therefore 441 = \frac{3n}{2}(n+7).$

$\Rightarrow n^2+7n-294=0$

$\therefore (n+21)(n-14)=0$

$\therefore n = 14$ **2**

(d) $2^3+2^4+\cdots+2^{13}$

$n=11,\ a=2^3;\ r=3.$

$\therefore S_{11} = 8(2^{11}-1)$

$= 8 \times 2047$

$= 16376.$ **2**

(e) $P(\text{even}) = \frac{1}{2}.$

$P(x<8) = \frac{7}{20}.$

$\therefore P(E)$ or $P(x<8)$

$= \frac{1}{2} + \frac{7}{20} - \frac{3}{20}.$

$= \frac{14}{20}$

$= 7/10$ **2**

(f) $S = 6\times 6 = 36$

sum > 7.

(2,6) (6,2) (3,5) (5,3)
(3,6) (6,3) (4,4), (4,5)
(5,4), (4,6) (6,4) (5,5)
(5,6) (6,5), (6,6).

$\therefore P(\text{sum}>7) = \frac{15}{36}$

$= \frac{5}{12}.$ **2**

(g) $(0\cdot 3)^3 = 0\cdot 027.$

(3).

(h)

$P(\text{at least 1 tail})$

$= 1 - Pr(\text{no tail})$

$= 1 - \frac{1}{2^n}$

2

[2] 2. (a) $$\frac{x^{2m+2}\times x^{3n+3}}{x^{5m}} = \frac{x^{2m+3n+5}}{x^{5m}},$$
$$= x^{3n-3m+5}.$$

Or (if you realised it was a typo)—
$$\frac{(x^{m+1})^2\times (x^3)^{m+1}}{x^{5m}} = \frac{x^{2m+2}\times x^{3m+3}}{x^{5m}},$$
$$= x^5.$$

[2] (b) $$2^{x-1} = 2^{1/2}\times 2^{-5},$$
$$x-1 = -4\tfrac{1}{2},$$
$$x = -3\tfrac{1}{2}.$$

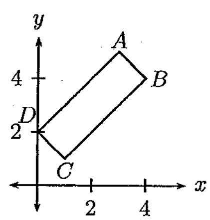
[2] (c) $$\frac{2^{n+2}+2^3}{2^{2n}+2^{n+1}} = \frac{2^2(2^n+2)}{2^n(2^n+2)},$$
$$= 2^{2-n}.$$

[2] (d) Slope $AB = \dfrac{0-{}^-2b}{2a-6a}$, $= \dfrac{2b}{-4a}$, $= -\dfrac{b}{2a}$.

Slope $BC = \dfrac{b-0}{0-2a}$, $= -\dfrac{b}{2a}$, $=$ Slope AB.

As B is common, ABC is a straight line.

[4] (e)



$m_{AB} = -1,$ so $m_{AB}\times m_{BC} = -1,$
$m_{BC} = 1,$ $m_{BC}\times m_{CD} = -1,$
$m_{CD} = -1,$ $m_{CD}\times m_{DA} = -1,$
$m_{DA} = 1.$ $m_{DA}\times m_{AB} = -1.$

$\therefore$ All vertices are right angles and $ABCD$ is a rectangle.

[4] (f) $\widehat{A}$ is common,
$A\widehat{B}C = A\widehat{D}E$ (corresponding angles, $BC\parallel DE$),
$\therefore\ \triangle ABC \mathbin{/\!/\!/} \triangle ADE$ (equiangular).

$$\frac{x}{x+7} = \frac{8}{13},\qquad \frac{y}{12} = \frac{13}{8},$$
$$13x = 8x+56,\qquad 2y = 39,$$
$$5x = 56,\qquad y = 19\tfrac{1}{2}$$
$$x = 11\tfrac{1}{5}$$

Question 3

a) $\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$

$= \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1}$

$= \lim_{x\to 1} (x+3)$

$= 4$ [2]

b) $y = x^2+2x+1$

(i) $y' = 2x+2$ [1]

(ii) At $(\frac{1}{2}, 2\frac{1}{4})$ $y' = 2(\frac{1}{2})+2$

$= 3$

$\therefore$ Gradient $= 3$ [1]

(iii) $m = 3 = \tan\alpha$

where α is the angle of inclination

$\therefore \alpha = \tan^{-1} 3$

$\doteq 71°34'$ [1]

(c) (i) $y = \frac{-7}{x+1}$

$= -7(x+1)^{-1}$

$y' = 7(x+1)^{-2} \times 1$

$= \frac{7}{(x+1)^2}$ [1]

(ii) $y = (x^2+x)^3$

$y' = 3(x^2+x)^2 \cdot \frac{d}{dx}(x^2+x)$

$= 3(x^2+x)^2 \times (2x+1)$

$= (6x+3)(x^2+x)^2$ [1]

$= 3x^2(2x+1)(x+1)^2$

(iii) $y = \frac{1}{\sqrt{3x^2+4}}$

$y' = \frac{-1}{(\sqrt{3x^2+4})^2} \times \frac{d}{dx}\sqrt{3x^2+4}$

$= \frac{-1}{3x^2+4} \times \frac{1}{2\sqrt{3x^2+4}} \cdot \frac{d}{dx}(3x^2+4)$

$= \frac{-6x}{2(3x^2+4)\sqrt{3x^2+4}}$

$= \frac{-3x}{(3x^2+4)^{3/2}}$ [2]

(d) $y = 5x\sqrt{4-x}$

$y' = 5\sqrt{4-x} + 5x \cdot \frac{-1}{2\sqrt{4-x}}$

$= 5\left[\sqrt{4-x} - \frac{x}{2\sqrt{4-x}}\right]$

At $x=3$ Grad. of tgt $= 5\left[1 - \frac{3}{2\times 1}\right]$

$= 5\left[-\frac{1}{2}\right]$

$= -\frac{5}{2}$

$\therefore$ Grad. of Normal $= \frac{2}{5}$ [3]

(e) $y = x^2-4x+3$

Function has a minimum.

$\therefore$ Max Value at boundary.

$y(1) = 1-4+3$

$= 0$

$y(4) = 16-16+3$

$= 3$

$\therefore$ Max Value $= 3$ [3]

QUESTION 4

(a) $y = 2x^3 - 3x^2 - 12x + 2$

(i) Stat. pts where $y' = 0$

$\therefore y' = 6x^2 - 6x - 12$

ie $y' = 6(x-2)(x+1) = 0$ 3

when $x = 2$ and $x = -1$

$\therefore (2, -18)$ and $(-1, 9)$

(ii) $y'' = 12x - 6$

At $x = 2$, $y'' = 18 > 0$

$\therefore$ MIN $(2, -18)$ 1

At $x = -1$, $y'' = -18 < 0$

$\therefore$ MAX $(-1, 9)$ 1

(iii) For P.O.I. $f''(x) = 0$ and there must be a change of concavity.

ie $f''(x)$ must change sign.

Now $f''(x) = 0$ when $x = \frac{1}{2}$

and

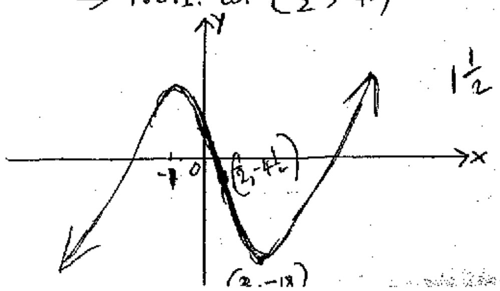
x	<	$\frac{1}{2}$	>
$f''(x)$	−	0	+

1

$\therefore$ at $x = \frac{1}{2}$ curve changes concavity from c.d. to c.u. 1

$\Rightarrow$ P.O.I. at $(\frac{1}{2}, -4\frac{1}{2})$

(iv) $1\frac{1}{2}$



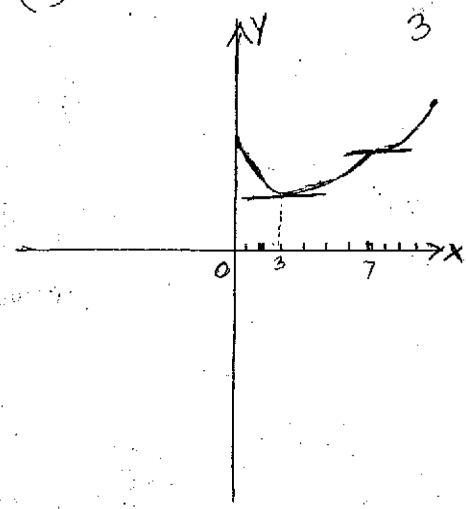
(b) $y = \frac{5}{x}$ 2

$\frac{dy}{dx} = -\frac{5}{x^2} < 0$

for all real x where $x \neq 0$

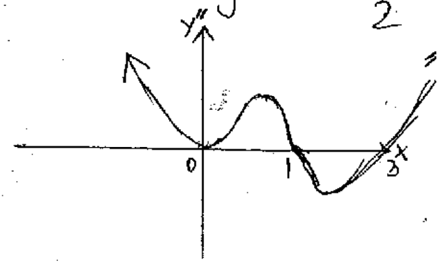
$\therefore y = \frac{5}{x}$ is decreasing for these values of x.

(c) 3



(d) $y'' = x^2(x-1)^3(x-3)$

c.u. when $y'' > 0$ 2



$\therefore$ c.u. when

$x < 1$ and $x > 3$

<u>Note</u>: $x \neq 0$

Question 5

(a)

$$2^x = 3^{x \quad 1}$$
$$\log_{10} 2^x = \log_{10} 3^{x \quad 1}$$
$$x \log_{10} 2 = (x \quad 1) \log_{10} 3$$
$$x \log_{10} 2 \quad x \log_{10} 3 = \quad \log_{10} 3$$
$$x(\log_{10} 2 \quad \log_{10} 3) = \log_{10} 3^{\quad 1} = \log_{10} 1/3$$
$$x(\log_{10} 2/3) = \log_{10} 1/3$$
$$x = \frac{\log_{10} 1/3}{\log_{10} 2/3} \quad 2 \;\; 71$$

(b)

$$x^2 + y^2 = 7xy \qquad x^2 + y^2 + 2xy = 9xy$$
$$x^2 + y^2 + 2xy = (x+y)^2 = 9xy$$
$$\log(x+y)^2 = \log 9xy$$
$$2\log(x+y) = \log 9 + \log x + \log y = \log 3^2 + \log x + \log y$$
$$2\log(x+y) = 2\log 3 + \log x + \log y$$
$$\log(x+y) = \log 3 + \tfrac{1}{2}\log x + \tfrac{1}{2}\log y$$

QED

(c) Distance

$$= 1 + 2 \quad (\tfrac{2}{3} \quad 1) + 2 \quad (\tfrac{2}{3} \quad \tfrac{2}{3}) + 2 \quad (\tfrac{2}{3} \quad (\tfrac{2}{3})^2) + \cdots$$
$$= 1 + 2 \quad \tfrac{2}{3} + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^3 + \ldots$$
$$= 1 + 2 \quad \frac{\frac{2}{3}}{1 \quad \frac{2}{3}} = 1 + 2 \quad \frac{\frac{2}{3}}{\frac{1}{3}} = 1 + 2 \quad 2 = 5$$

(d) The first \$3000 would earn $3000(1 \;\; 06)^{35}$, the next \$3000 would earn $3000(1 \;\; 06)^{34}$ and so on until the start of the 35th year where the last \$3000 would earn $3000(1 \;\; 06)$.

So the total investment is worth

$$S = 3000(1 \;\; 06)^{35} + 3000(1 \;\; 06)^{34} + \cdots + 3000(1 \;\; 06)$$
$$S = 3000 \quad (1 \;\; 06) + 3000(1 \;\; 06)^2 + \cdots + 3000(1 \;\; 06)^{35}$$
$$= 3000 \quad \frac{1 \;\; 06 \left((1 \;\; 06)^{35} \quad 1 \right)}{1 \;\; 06 \quad 1}$$
$$= \$354362 \;\; 60$$

(e) Let $\$M$ be the monthly repayment, let $\$A_n$ be the amount owing after n months.

12% pa = 1% per month, 10 years = 120 months

$$A_1 = 75000(1\ \ 01) \quad M$$
$$A_2 = A_1(1\ \ 01) \quad M$$
$$= 75000(1\ \ 01)^2 \quad M(1 + 1\ \ 01)$$
$$A_3 = A_2(1\ \ 01) \quad M$$
$$= 75000(1\ \ 01)^3 \quad M(1 + 1\ \ 01 + 1\ \ 01^2)$$
$$A_n = 75000(1\ \ 01)^n \quad M(1 + 1\ \ 01 + \ldots + 1\ \ 01^{n\ \ 1})$$
$$A_{120} = 75000(1\ \ 01)^{120} \quad M(1 + 1\ \ 01 + \ldots + 1\ \ 01^{119})$$

$$\text{Let } S_{120} = 1 + 1\ \ 01 + \ldots + 1\ \ 01^{119}$$
$$= \frac{1\ \ 01^{120} \quad 1}{1\ \ 01 \quad 1} = 100(1\ \ 01^{120} \quad 1)$$
$$A_{120} = 0$$
$$M = \frac{75000(1\ \ 01)^{120}}{S_{120}} = 1076\ \ 03$$

So the monthly repayment is $\$1076\ \ 03$